Global Transformation of Rotation Matrices to Euler Parameters

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Introduction

Two widely used forms of attitude parameterization are the rotation matrix form and the Euler parameter form. Each form is globally nonsingular and numerically ideal because rotation group algebra in terms of each involves orthogonal transformations. The transformation of one form to the other is of both practical and theoretical interest. The transformation of Euler parameters to rotation matrices can be expressed globally as a single matrix equation, but the same is not true of the opposite transformation.

The transformation of rotation matrices to Euler parameters was solved by Klumpp [1] and Shepperd [2], but their transformation algorithms are not truly global. The Klumpp algorithm is not truly global because for certain attitudes it produces indeterminate forms that must be resolved by additional steps. The Shepperd algorithm is not truly global because it uses different nonglobal transformations for different regions of the attitude state space.

This paper presents the first global algorithm for transforming rotation matrices to Euler parameters. Although it has no apparent computational or numerical advantage over the known algorithms, it provides insight into the relationship between the two forms. It makes use of the singular value decomposition (SVD), a numerically ideal algorithm involving orthogonal transformations. The subsequent sections review attitude parameterization, establish an analytical framework, and finally present the new transformation algorithm.

Attitude Parametrization

According to Euler's theorem, the attitude of a rigid body can be reached from any arbitrary reference attitude by a rotation about an axis referred to as the Euler axis or eigenaxis. Suppose that two Cartesian coordinate frames are specified: an arbitrary reference frame, and a body frame that is fixed with respect to the rigid body. The reference attitude is defined such that the body coordinate frame is aligned with the reference coordinate frame. Let $a \in \mathbb{R}^3$ be the identical body and reference coordinates of a unit vector aligned with the eigenaxis, and let $\phi \in \mathbb{R}$ be the angle of rotation about the eigenaxis, defined in a right-hand sense. Let $R \in \mathbb{R}^{3\times 3}$ and $\beta \in \mathbb{R}^4$ be the rotation matrix and the Euler parameters, respectively, corresponding to the attitude of the body frame relative to the reference frame. Also, let $\varepsilon \in \mathbb{R}^3$ be the first three Euler parameters, and let $\eta \in \mathbb{R}$ be the fourth.

The rows of R are the reference coordinates of unit vectors aligned with the corresponding body axes; the columns of R are the body coordinates of unit vectors aligned with the corresponding reference axes. Premultiplication by R transforms the reference coordinates of a vector to the body coordinates of the same vector; premultiplication by R^T does the opposite. The rotation matrix R is expressed in terms of the eigenaxis coordinates a and the rotation angle ϕ as

$$\begin{aligned} R &= \exp(-\phi a \times) & (1) \\ &= (\cos\phi) I + (1-\cos\phi) a a^T - (\sin\phi) a\times & (2) \end{aligned}$$

where the skew-symmetric cross-product operator is defined for an arbitrary three-component variable as follows:

$$\kappa \equiv \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} \quad \rightarrow \quad \kappa\times \equiv \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix}$$

Because $\kappa \times \kappa = 0$, $\kappa\times$ is singular and κ is in its nullspace. Because the reference frame and the body frame are both orthogonal, so is the rotation matrix R, hence $R^T R = R R^T = I$, $R^{-1} = R^T$, and all three singular values of R are 1.

The Euler parameters β are expressed by definition in terms of the eigenaxis coordinates a and the rotation angle ϕ as

$$\begin{aligned} \varepsilon &= \sin(\phi/2)\, a & (3) \\ \eta &= \cos(\phi/2) & (4) \end{aligned}$$

The Euler parameters (which are equivalent to the coefficients of a unit quaternion) have unit norm by definition, hence $\|\beta\|^2 = \beta^T\beta = \varepsilon^T\varepsilon + \eta^2 = 1$. The Euler parameters do not uniquely parametrize attitude because if the signs of all four parameters are changed they still correspond the same physical attitude (they correspond mathematically to an odd number of complete revolutions about the eigenaxis).

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Transformation of Euler parameters to rotation matrices is expressed as [3, 4]

$$R = (\eta^2 - \varepsilon^T \varepsilon) I + 2\varepsilon\varepsilon^T - 2\eta\varepsilon\times \tag{5}$$

Analysis

Singular value decomposition [5, 6] is the factorization of any matrix into a product of the form USV^T, where U and V are square and orthogonal, and S is diagonal, has the same dimensions as the original matrix, and contains the nonnegative singular values in decreasing order along the diagonal. The columns of U corresponding to the nonzero singular values form a basis for the range space of the original matrix. The columns of V corresponding to the singular values equal to zero form a basis for the null space of the original matrix.

According to Euler's theorem, the eigenaxis coordinates a remain invariant under a coordinate transformation by any related rotation matrix of the form $\exp(-\alpha a\times)$ for any rotation angle $\alpha \in I\!\!R$. For the particular case in which $\alpha = \phi$, because $R = \exp(-\phi a\times)$ this invariance is expressed as

$$Ra = R^T a = a \tag{6}$$

The eigenaxis coordinates thus constitute an eigenvector of the related rotation matrix, with a corresponding eigenvalue of 1. The other eigenvalues of R are $\exp(\pm i\phi)$. Equation 6 is equivalent to

$$(R - I)a = (R^T - I)a = 0 \tag{7}$$

Both $R - I$ and $R^T - I$ are therefore singular, and a is in their nullspace.

Singular value decompositions of each are of the form

$$R - I = USV^T \tag{8}$$

$$R^T - I = VSU^T \tag{9}$$

where $S \in I\!\!R^{3\times3}$ is diagonal and contains the nonnegative singular values in decreasing order along the diagonal, and where $U, V \in I\!\!R^{3\times3}$ are orthogonal, hence $U^TU = UU^T = V^TV = VV^T = I$. Let u_i and v_i be the ith columns of U and V, respectively, and let s_i be the ith singular value, so that $USV^T = \Sigma_{i=1}^3 s_i u_i v_i^T$. Because $R - I$ and $R^T - I$ are singular, their last singular values must be zero, hence $s_3 = 0$. Also, because the columns of U and V corresponding to the singular value of zero form a basis for the null space, $u_3 = v_3 = \pm a$. The matrices U and V, which are rotation matrices themselves, therefore correspond to orthogonal frames that are related by some as yet unknown rotation angle about their common third axis, which is the eigenaxis.

In the basic singular value decomposition, the matrices U and V can have determinants of ± 1. The singular value decomposition of a matrix is not unique, because if USV^T is a valid singular value decomposition, so is $(-U)S(-V)^T$, among others. Note that $\det(kM) = k^n \det M$, where k is an arbitrary scalar and M is an arbitrary $n \times n$ matrix. Because $k = -1$ and $n = 3$ in this case, $\det(-U) = -\det U$ and $\det(-V) = -\det V$. It can therefore be specified without loss of generality that $\det U = \det V = 1$, so that both U and V are right-handed orthogonal rotation matrices.

Because $R = \exp(-\phi a\times)$, it is clear that

$$\exp(\alpha a\times)(R - I)\exp(-\alpha a\times) = R - I \tag{10}$$

for all $\alpha \in I\!\!R$. By substituting equation 8 into equation 10 and rearranging, it also becomes clear that

$$[\exp(\alpha a\times)U]\; S\; [\exp(\alpha a\times)V]^T = USV^T \tag{11}$$

for all $\alpha \in I\!\!R$. Therefore, rotation of the frames corresponding to U and V each by the same arbitrary angle about the eigenaxis produces another valid singular value decomposition. The first two singular values must therefore be equal. Also, because

$$[\exp(\alpha a\times)U]^T\; [\exp(\alpha a\times)V] = U^TV \tag{12}$$

for all $\alpha \in I\!\!R$, the product on the left side is invariant with respect to α. The product U^TV is therefore a rotation matrix corresponding to a rotation by some as yet unknown angle about the eigenaxis. The angle of rotation will now be determined.

Let A be a right-handed orthogonal rotation matrix defined as

$$A = [\; a_1 \quad a_2 \quad a \;] \in I\!\!R^{3\times3} \tag{13}$$

where the third column contains the eigenaxis coordinates. Because A is orthogonal, $A^TA = AA^T = I$, which implies that $a_1^Ta_1 = a_2^Ta_2 = a^Ta = 1$ and $a^Ta_1 = a^Ta_2 = a_1^Ta_2 = 0$. Also, $\det A = 1$, which implies that $a_2 = a \times a_1 = -a_1 \times a_2$ and $a_1 = a_2 \times a = -a \times a_2$. It has been shown above that all possible U and V matrices are parameterized by a rotation of the frame corresponding to A about the eigenaxis by some arbitrary angle. Therefore, let

$$U = \exp(\alpha a\times)A \tag{14}$$

$$V = \exp(\gamma a\times)A \tag{15}$$

where the relationship between angles α and γ is to be determined. With the previous identities and equation 2, it has been determined that

$$U = [\; \mathrm{c}\alpha a_1 - \mathrm{s}\alpha a_2 \quad \mathrm{c}\alpha a_2 + \mathrm{s}\alpha a_1 \quad a \;] \tag{16}$$

$$V = [\; \mathrm{c}\gamma a_1 - \mathrm{s}\gamma a_2 \quad \mathrm{c}\gamma a_2 + \mathrm{s}\gamma a_1 \quad a \;] \tag{17}$$

where c and s are short for cosine and sine. Also, let

$$S = 2 \mid \mathrm{c}(\alpha - \gamma) \mid \mathrm{diag}\{1, 1, 0\} \tag{18}$$

By performing the matrix multiplication and simplifying, it has been established that

$$USV^T = 2 \mid \mathrm{c}(\alpha-\gamma) \mid [\mathrm{c}(\alpha-\gamma)(I - aa^T) - \mathrm{s}(\alpha-\gamma)a\times] \tag{19}$$

The following identities have been used: $a_2 a_1^T - a_1 a_2^T = a\times$, $a_1 a_2^T - a_2 a_1^T = -a\times$, $\mathrm{c}\alpha\mathrm{c}\gamma + \mathrm{s}\alpha\mathrm{s}\gamma = \mathrm{c}(\alpha - \gamma)$, and $\mathrm{s}\alpha\mathrm{c}\gamma - \mathrm{c}\alpha\mathrm{s}\gamma = \mathrm{s}(\alpha - \gamma)$.

By substituting equations 19 and 2 into equation 8, squaring both sides, and simplifying, it has been established that

$$\alpha - \gamma = \phi/2 \pm \pi/2 \tag{20}$$

In addition to the previous identities, the following identities have been used: $2\mathrm{c}^2\theta = 1 + \mathrm{c}(2\theta)$, $2\mathrm{s}^2\theta = 1 - \mathrm{c}(2\theta)$, and $2\mathrm{s}\theta\mathrm{c}\theta = \mathrm{s}(2\theta)$, with $\theta = \alpha - \gamma$. It then follows from equations 16 and 17 and the previous identities that

$$U^T V = \begin{bmatrix} \sin(\phi/2) & \cos(\phi/2) & 0 \\ -\cos(\phi/2) & \sin(\phi/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{21}$$

for the case in which $\alpha - \gamma = \phi/2 + \pi/2$. For the other case, in which $\alpha - \gamma = \phi/2 - \pi/2$, the signs of the upper left two-by-two block are all changed. Both the plus and the minus sign are valid in equation 20 because if the signs of all four Euler parameter are changed, they still correspond to the same physical attitude.

Transformation Algorithm

The algorithm for globally transforming rotation matrices to Euler parameters can now be stated as follows. For the rotation matrix R, perform a singular value decomposition of $R - I$, where I is the identity matrix, to obtain

$$R - I = USV^T \tag{22}$$

Let $u_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ be the ith columns of U and V, respectively, and let $s_i \in \mathbb{R}$ be the ith singular value. Then $\det(U)u_3 = \det(V)v_3 = \pm a$, where a is the eigenaxis coordinates. Also, $u_1^T v_1 = u_2^T v_2 = \sin(\phi/2)$ and $u_1^T v_2 = -u_2^T v_1 = \cos(\phi/2)$, where ϕ id the angle of rotation about the eigenaxis. The transformation can therefore be expressed in several possible ways, such as

$$\varepsilon = \det(V) v_3 \, u_1^T v_1 \tag{23}$$

$$\eta = u_1^T v_2 \tag{24}$$

The multiplication by $\det(V)$ above should not actually be performed, but rather the signs should simply be reversed if $\det(V) = -1$.

The algorithm may fail if $R = I$ because the singular-value decomposition of $R - I = 0$ is not well defined. Fortunately, however, this case can easily be detected by the algorithm and the transformation is then trivial. Thus, the only exceptional case is that if $s_1 = 0$, set $\varepsilon = 0$ and $\eta = 1$. No numerical problems occur for cases where R is very close to but not equal to I. The existence of this exception seems to mean that the algorithm is not truly global, but since the exception is only a single point that is easily detectable and requires no alternative computation, the algorithm is still considered global.

If a sequence of Euler parameters approximating a continuous function of time is required, then the appropriate signs must be chosen for the Euler parameters. This can be done by taking the inner product of the current Euler parameters with the previous ones, and changing the signs of the current ones if that inner product is negative.

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