Attitude Control with Realization of Linear Error Dynamics

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Abstract: An attitude control law is derived to realize linear error dynamics, with the attitude error properly defined in terms of rotation group algebra (rather than vector algebra). Euler parameters are used in the rotational dynamics model because they are globally nonsingular, but only the minimal three Euler parameters are used in the error dynamics model because they have no nonlinear mathematical constraints to prevent the realization of linear error dynamics. The control law is singular only when the attitude error angle is exactly π rad about any eigenaxis, and a simple intuitive modification at the singularity allows the control law to be used globally. The forced error dynamics are nonlinear but stable. Numerical simulation tests show that the control law performs robustly for both initial attitude acquisition and attitude control.

Introduction

The conventional approach to attitude control is based on linear control theory. The nonlinear rotational dynamics model is adapted to linear control theory by approximating it with a set of linear dynamic models that are tangent to it at selected design states. The tangent models are in the form of transfer functions or state-space models, depending on whether the frequency-domain or time-domain version of linear control theory is to be used. Linear control theory is applied to each tangent model separately, and the control parameters are scheduled as required in flight. This approach is used for virtually all currently operational aircraft and for some spacecraft. It is also still widely used in current research and development.

Linear control theory was perhaps the only practical alternative when onboard computers were severely limited, but for the following reasons it may not be the best alternative now. First, the rigid-body dynamics and the moment generation are coupled together in the linear tangent models into a single abstract mathematical model without clear physical meaning. Thus a mere change in the inertia tensor, for example, necessitates in principle a complete resynthesis of the control parameters. Furthermore, only linear moment generation models can be used. Advanced nonlinear aerodynamic models for aircraft cannot be used directly. Also, the attitude error is defined in terms of vector algebra, which is valid only for small angular displacements. Finally, stability away from the design states is difficult or impossible to mathematically guarantee, especially if the dynamic effects of the parameter scheduling are properly considered.

Attitude control theory was first introduced by Meyer [1, 2]; other approaches were introduced later by Mortensen [3], Dwyer [4, 5], Wie et al. [6, 7], and Slotine and Li [8]. In contrast to linear control theory, attitude control theory applies directly to the nonlinear rotational dynamics. In each approach, the rigid-body dynamics and the moment generation are decoupled into separate mathematical models, and nonlinear moment generation models of arbitrary complexity and sophistication can be used directly. These approaches can be divided into the two major categories outlined below, and this paper introduces a third, which incorporates features of the other two but avoids their main problems.

Dwyer, and Slotine and Li, each used nonlinear transformations to realize an exact linear model of the rotational dynamics, to which linear control theory can be applied. The state variables are some minimal (three-parameter) attitude form and its derivative with respect to time. Dwyer used the minimal three Euler parameters, and Slotine and Li used Euler angles. The main problem here is that a linear model of rotational dynamics cannot be global because the transformations required to realize it violate the topology of the attitude state space. Thus the attitude control law is singular at certain attitudes, even with no attitude error. The singularities could be avoided in practice by reorienting the coordinate frame online as necessary, but that would add complexity and violate linearity. Furthermore, to maintain linearity the attitude error must be defined in terms of vector algebra rather than rotation group algebra. Such a definition has no geometric meaning, and is inappropriate in general, because attitude is not a vector.

Meyer, Mortensen, and Wie used Lyapunov control theory. In Lyapunov control theory, a control law and an associated Lyapunov function are postulated by intuition, and stability is determined by analysis of the Lyapunov function. The control law consists of feedback of a linear combination of 1) the body coordinates of the angular rate error and 2) some minimal attitude error form that is appropriately defined in terms of rotation group

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algebra. Feedback of attitude and angular rate errors are intuitively analogous to feedback of position and velocity errors to provide stiffness and damping in a linear position controller. Mortensen and Wie used Euler parameters, whereas Meyer used rotation matrices. For each case, it is shown in this paper that the resulting unforced error dynamics are nonlinear, but are approximately linear for small errors. A disadvantage of Lyapunov control theory is that the control law is based on intuition rather than fundamental principles. Another disadvantage is that important concepts such as damping and loop bandwidth are not well defined as in linear control theory.

In the approach presented in this paper, an attitude control law is derived to realize linear unforced error dynamics with the attitude error defined in terms of rotation group algebra (rather than vector algebra). Euler parameters are used in the rotational dynamics model because they are globally nonsingular, but only the minimal three Euler parameters are used in the error dynamics model because they have no nonlinear mathematical constraints to prevent the realization of linear error dynamics. The control law is singular only when the attitude error angle is exactly π rad about any eigenaxis, and a simple intuitive modification at the singularity allows the control law to be used globally. Exact linearity is realized for the first time with an appropriate definition of attitude error.

Rotational Dynamics

The classical rotational dynamics of a rigid body [9, 10] are reviewed in this section. Two right-handed, three-dimensional cartesian coordinate frames are used: an inertial frame that is fixed with respect to inertial space, and a body frame that is fixed with respect to the body of the vehicle. The symbol $\mathbb{R}$ represents the real numbers; I represents an identity matrix of appropriate size; the superscript T indicates transposition; a dot over a variable indicates differentiation with respect to time; and the skew-symmetric cross-product operator is defined for an arbitrary three-component variable as follows:

$$\kappa \equiv \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} \rightarrow \kappa\times \equiv \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix} \qquad (1)$$

Because $\kappa \times \kappa = 0$, $\kappa\times$ is singular and κ is in its null space.

Kinematics

According to Euler's theorem, any attitude can be reached from any reference attitude by a pure rotation about an axis referred to as the Euler axis or eigenaxis. Let the reference attitude be defined such that the body frame is aligned with the inertial frame. Let a represent the identical inertial and body coordinates of a unit vector aligned with the eigenaxis, and let ϕ be the angle of rotation about the eigenaxis, defined in a right-handed sense. For $|\phi| < \pi$ the nonlinear rotational kinematics are described by

$$\dot{\phi} = a^T \omega \qquad (2)$$

$$\dot{a} = [I - \cot(\phi/2) a\times](a \times \omega)/2 \qquad (3)$$

where ω represents the body coordinates of the angular rate of the vehicle relative to inertial space, defined in a right-handed sense.

Direction Cosines

Let $R \in \mathbb{R}^{3\times3}$ be the rotation matrix corresponding to the attitude of the body frame relative to the inertial frame. The rows of R are the inertial coordinates of unit vectors aligned with the corresponding body axes; the columns of R are the body coordinates of unit vectors aligned with the corresponding inertial axes. The i, jth element of R, referred to as a direction cosine, is the cosine of the angle between the ith axis of the body frame and the jth axis of the inertial frame. Premultiplication by R transforms the inertial coordinates of a vector to the body coordinates of the same vector; premultiplication by R^T does the opposite. Because the inertial and body frames are orthogonal, the rotation matrix is also orthogonal, so

$$R^T R = R R^T = I \qquad (4)$$

and all three of the singular values of R are 1.

The rotation matrix R is expressed in terms of the eigenaxis coordinates a and the rotation angle ϕ as

$$R = \exp(-\phi a\times) \qquad (5)$$

$$= (\cos\phi) I + (1 - \cos\phi) a a^T - (\sin\phi) a\times \qquad (6)$$

Because $Ra = R^T a = a$, a is an eigenvector of both R and R^T, with a corresponding eigenvalue of 1. The other eigenvalues of R are $\exp(\pm i\phi)$.

The rotational kinematics are described in terms of rotation matrices by

$$\dot{R} = -\omega \times R \qquad (7)$$

Euler Parameters

Let β represent the Euler parameters corresponding to the attitude of the body frame relative to the inertial frame. Also, let ε represent the first three Euler parameters and let η be the fourth, so that

$$\beta \equiv \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \in \mathbb{R}^4, \quad \varepsilon \in \mathbb{R}^3, \quad \eta \in \mathbb{R} \qquad (8)$$

The Euler parameters, which are equivalent to the coefficients of a unit quaternion, have unit norm by definition; hence

$$\|\beta\|^2 = \beta^T \beta = \varepsilon^T \varepsilon + \eta^2 = 1 \qquad (9)$$

The Euler parameters β are expressed in terms of the eigenaxis coordinates a and the rotation angle ϕ as

$$\varepsilon = \sin(\phi/2) a \qquad (10)$$

$$\eta = \cos(\phi/2) \qquad (11)$$

The Euler parameters do not uniquely parameterize attitude because if the signs of all four parameters are changed they still correspond to the same physical attitude (they correspond mathematically to an odd number of complete revolutions about the eigenaxis). The nonuniqueness of the Euler parameters can be resolved without singularities, however, by making an arbitrary initial choice and then simply requiring that the parameters be continuous in time.

The rotational kinematics are described in terms of Euler parameters by

$$\dot{\beta} = Q\beta/2 \tag{12}$$

where Q is a transformation matrix defined as

$$Q \equiv \begin{bmatrix} -\omega\times & \omega \\ -\omega^T & 0 \end{bmatrix} \in I\!R^{4\times4} \tag{13}$$

Because $QQ^T = Q^TQ = (\omega^T\omega)I$, Q is orthogonal for $\omega \neq 0$, and all four of its singular values are $\| \omega \|$. The eigenvalues of Q occur in pairs at each of the points $\pm i \, \|\omega\|$. Equation 12 is equivalent to

$$\dot{\beta} = U\omega/2 \tag{14}$$

where U is a transformation matrix defined as

$$U \equiv \begin{bmatrix} T \\ -\varepsilon^T \end{bmatrix} \in I\!R^{4\times3} \tag{15}$$

and T is a transformation matrix defined as

$$T \equiv \eta I + \varepsilon\times \in I\!R^{3\times3} \tag{16}$$

Equation 14 is equivalent to the two equations

$$\dot{\varepsilon} = T\omega/2 \tag{17}$$

$$\dot{\eta} = -\varepsilon^T\omega/2 \tag{18}$$

Because $U^TU = T^TT + \varepsilon\varepsilon^T = I$, U is column orthogonal and all three of its singular values are 1. Also, because $U^T\beta = \dot{U}^T\dot{\beta} = 0$, the inverse rotational kinematics are described by

$$\omega = 2U^T\dot{\beta} \tag{19}$$

$$\dot{\omega} = 2U^T\ddot{\beta} \tag{20}$$

The transformation matrix T defined in equation 16 has properties and relationships that greatly simplify the attitude control law to be derived. Because

$$T^{-1} = T^T + \varepsilon\varepsilon^T/\eta \tag{21}$$

it is apparent that T is singular at $\eta = 0$ or $\phi = \pm\pi$ rad. Also, from the definition of T it is obvious that

$$T\varepsilon = T^T\varepsilon = \eta\varepsilon \tag{22}$$

Hence ε is an eigenvector of both T and T^T, with a corresponding eigenvalue of η or $\cos(\phi/2)$. This fact will be used extensively in the development of the attitude control law. The other eigenvalues of T are $\eta \pm i \, \| \varepsilon \|$, or $\exp(\pm i\phi/2)$. The singular values of T are 1, 1, and $|\eta|$. It has been discovered in this study that the transformation matrix T and the rotation matrix R are related according to

$$R = T^{-1}T^T = T^TT^T + \varepsilon\varepsilon^T \tag{23}$$

This fact will also be used in the development of the attitude control law. Expansion of equation 23 produces the known form

$$R = (\eta^2 - \varepsilon^T\varepsilon)I + 2\varepsilon\varepsilon^T - 2\eta\varepsilon\times \tag{24}$$

Note also that

$$R\varepsilon = R^T\varepsilon = \varepsilon \tag{25}$$

Hence ε is also an eigenvector of R and R^T, with a corresponding eigenvalue of 1.

Kinetics

The total angular momentum of the vehicle consists of the angular momentum of the rigid body of the vehicle, plus the internal angular momentum stored in the vehicle. For spacecraft, the internal angular momentum is stored in a momentum storage system, which consists of a set of reaction wheels or control moment gyros. For aircraft, if the propulsion system is asymmetric, a net internal angular momentum is stored in rotating engine spools, propellers, or rotors. Let H represent the body coordinates of the total angular momentum relative to inertial space. Then

$$H = J\omega + h \tag{26}$$

where h represents the body coordinates of the internal angular momentum relative to the *vehicle*, and J is the vehicle inertia tensor with respect to the body frame, *including* the inertia of the momentum storage system. (If J had been defined to *exclude* the inertia of the momentum storage system, then equation 26 would be true if h represented the body coordinates of the internal angular momentum relative to *inertial space*. The convention adopted here is more convenient, however, because the internal angular momentum relative to the *vehicle* can be measured.)

Let M represent the body coordinates of the applied moment. According to classical dynamics, the inertial coordinates of the applied moment are equal to the rate of change of the inertial coordinates of the inertial angular momentum. Thus, $R^TM = d(R^TH)/dt = R^T\dot{H} + \dot{R}^TH$, where R^TM and R^TH represent the inertial coordinates of the moment and the angular momentum, respectively. By premultiplying by R, and using equation 7, the rotational kinetics equation

$$M = \dot{H} + \omega \times H = J\dot{\omega} + \dot{h} + \omega \times (J\omega + h) \tag{27}$$

is derived. It has been assumed that the inertia of the momentum storage system is invariant with respect to the body frame. It has also been assumed for now that the moment and the momentum transfer rate are directly controllable.

Continuous mass ejection, such as the exhaust of burned fuel or the release of internal gases for control, can be accounted for as an applied moment. An additional term $\dot{J}\omega$ could be added to the right side of equation 27 to account for the rate of change of the inertia tensor, but it is usually negligible.

Attitude Control

Superscripts on variables are defined as follows:

c	open-loop feedforward command
$*$	closed-loop feedforward/feedback command
e	rotational error

Variables without superscripts represent actual values, as before. A superscript on a matrix indicates that the superscript applies to each of the variables in the matrix.

A simplified diagram of the attitude controller is shown in Figure 1. Accurate estimates of attitude and angular rate are assumed to be available. Euler parameters are used for attitude. Although Euler parameters are not directly measurable, any direct measurement of attitude can be transformed into them. Attitude is not directly measured in an inertial navigation system, so Euler parameters can be used as well as anything else in the numerical integration of angular rate measurements.

A feedforward command generator is required to accept the raw attitude or angular rate commands and to generate a consistent set of feedforward command variables $\{\beta^c, \omega^c, \dot{\omega}^c\}$ for attitude, angular rate, and angular acceleration, respectively. In the most basic form, the feedforward command generator consists of an inverse rotational dynamics model, but in that case the raw commands must be suitably differentiable. Alternatively, the feedforward command generator can be designed to precondition the raw commands if that is considered appropriate for engineering reasons. Command preconditioning might include filtering of noise on analog real-time commands, for example. It could also include filtering and rate limiting of the raw commands to 1) prevent actuator saturation; 2) attenuate actuator stress, energy consumption, or excitation of structural vibration modes; or 3) avoid dangerous flight conditions. Because the feedforward command generator is specific to each application, it is outside the scope of this paper.

Attitude Error

Because attitude has only three degrees of freedom, a minimal attitude parameterization form must have exactly three components. It has been shown [11], however, that a globally continuous mapping of the three-dimensional rotation group onto $\mathbb{R}^3$ does not exist. A minimal attitude form that is globally nonsingular therefore does not exist either.

Consider Euler parameters, for example. The magnitude of the fourth Euler parameter is dependent, according to the unit-norm constraint, on the first three parameters, but its sign is independent. The actual information content thus consists of three real components and a sign. But if all four of the Euler parameters have their signs changed, they still correspond to the same physical attitude. The fourth (or any other) Euler parameter can therefore be arbitrarily constrained to be positive, and the other three parameters will constitute a minimal parameterization of attitude. A singularity will then occur, however, when the rotation angle crosses $\pm\pi$ rad about any eigenaxis, because then the signs of all four parameters must be changed to keep the fourth parameter positive.

The nonminimal or over-parameterized nature of globally nonsingular attitude forms is such that nonlinear constraints exist among their elements. These nonlinear constraints make linear dynamics impossible to realize in terms of all the elements simultaneously. However, a globally nonsingular form is not required for the attitude error if the error angle does not approach π rad about any eigenaxis. A minimal attitude error form can then be used so that linear error dynamics can be realized.

In terms of rotation matrices, the relationship among the attitude, the attitude command, and the attitude error, is $R = R^e R^c$, or $R^e = RR^{cT}$ In terms of Euler parameters, the relationship is $\beta = V^c\beta^e = W^{eT}\beta^c$, where V and W are transformation matrices defined as

$$V \equiv \begin{bmatrix} T & \varepsilon \\ -\varepsilon^T & \eta \end{bmatrix} \equiv \begin{bmatrix} U & \beta \end{bmatrix} \in \mathbb{R}^{4\times 4} \quad (28)$$

$$W \equiv \begin{bmatrix} T & -\varepsilon \\ \varepsilon^T & \eta \end{bmatrix} \in \mathbb{R}^{4\times 4} \quad (29)$$

Because $V^TV = VV^T = W^TW = WW^T = I$, both V and W are orthogonal and all four of the singular values of each are 1. The eigenvalues of V and W occur in pairs at each of the points $\eta \pm i \parallel \varepsilon \parallel$, or $\exp(\pm i\phi/2)$. The error Euler parameters are determined from

$$\beta^e = V^{cT}\beta \quad (30)$$

This is equivalent to the two equations

$$\varepsilon^e = U^{cT}\beta \quad (31)$$

$$\eta^e = \beta^{cT}\beta \quad (32)$$

where ε^e is a minimal attitude error form. Note that ε^e is scaled version of the body coordinates of the error eigenaxis and is valid only for error angles in the range $0 \leq \phi^e < \pi$ rad about any eigenaxis. Because $\parallel \varepsilon^e \parallel = \sin(\phi^e/2)$ and $\eta^e = \cos(\phi^e/2)$, for small errors the magnitude of ε^e is approximately equal to *half* of the error angle, and η^e is approximately 1. For perfect tracking of the attitude command, $\beta = \beta^c$, $\eta^e = 1$, and $\varepsilon^e = 0$.

Unforced Error Dynamics

The unforced error dynamics are the response of the attitude and angular-rate errors, given their initial conditions, in the ideal case when the error forcing function is zero. This occurs when the closed-loop moment commands are realized exactly. Details of the development of the unforced error dynamics are given in the appendix.

By differentiating equation 31 with respect to time, and simplifying based on equations 9, 14, 22, and 23, it has been determined in this study that the unforced error kinematics are described by

$$\dot{\varepsilon}^e = T^e\omega^e/2 \tag{33}$$

$$\ddot{\varepsilon}^e = T^e\dot{\omega}^e/2 - \varepsilon^e(\omega^{eT}\omega^e)/4 \tag{34}$$

where

$$\omega^e = \omega - R^e\omega^c \tag{35}$$

$$\dot{\omega}^e = \dot{\omega} - R^e\dot{\omega}^c - \omega \times \omega^e \tag{36}$$

Equation 33 could have been anticipated because it is of the same form as equation 17. However, the definition of angular rate error in equation 35 has been determined in the derivation of equation 33, and equation 23 has been used to determine that definition. Whereas ω^c represents the coordinates of the angular rate command resolved in the *commanded* body frame, the term $R^e\omega^c$ in equation 35 represents the coordinates of the angular rate command resolved in the *actual* body frame, so the subtraction in equation 35 is in terms of consistent coordinates. The feedforward angular rate and acceleration commands are

$$\omega^c = 2U^{cT}\dot{\beta}^c \tag{37}$$

$$\dot{\omega}^c = 2U^{cT}\ddot{\beta}^c \tag{38}$$

which are consistent with equations 19 and 20.

An attitude control law will be derived to realize unforced error dynamics described by

$$\ddot{\varepsilon}^e = f(\varepsilon^e, \dot{\varepsilon}^e, t) \tag{39}$$

where $\dot{\varepsilon}^e = T^e\omega^e/2$, and f represents an arbitrary linear or nonlinear function. The error dynamics to be realized could also be expressed in terms of any other minimal attitude error form, such as Euler angles, for example. Euler parameters were chosen because they have ideal numerical properties and because a series of elegant simplifications occurs in the derivation of the control law.

An important special case is that in which the unforced error dynamics are linear and time-invariant, as when described by

$$\ddot{\varepsilon}^e + c_1\dot{\varepsilon}^e + c_0\varepsilon^e = 0 \tag{40}$$

where the damping coefficient c_1 and the stiffness coefficient c_0 are constant scalar design parameters. The poles are the roots of the characteristic equation $s^2+c_1s+c_0 = 0$. They can easily be placed anywhere in the s-plane. The coefficients c_0 and c_1 could be generalized to matrix coefficients C_0 and C_1 if desired, but the derivation of the control then becomes much more complicated. In that case, the poles would be the roots of the characteristic equation $\det(Is^2 + C_1s + C_0) = 0$.

For some applications integral-error feedback may be necessary to drive the steady-state attitude error to zero in response to a steady bias moment. For that case, the unforced error dynamics are still linear and time-invariant, but are now described by

$$\ddot{\varepsilon}^e + c_1\dot{\varepsilon}^e + c_0\varepsilon^e + c_i\int \varepsilon^e dt = 0 \tag{41}$$

where the integral-error coefficient c_i is a constant scalar design parameter, and the other symbols are as previously defined. The poles are the roots of the characteristic equation $s^3 + c_1s^2 + c_0s + c_i = 0$.

By substituting equations 33 and 34 into equation 40 and simplifying based on equations 9, 14, 22, and 23, it has been established in this study that

$$\begin{aligned}&\ddot{\varepsilon}^e + c_1\dot{\varepsilon}^e + c_0\varepsilon^e = \\ &T^e\left[\dot{\omega}^e + c_1\omega^e + 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e\right]/2\end{aligned} \tag{42}$$

After this is premultiplied by $(T^e)^{-1}$, it is clear that equation 40 is equivalent to

$$\dot{\omega}^e + c_1\omega^e + 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e = 0 \tag{43}$$

away from the singularity at $\eta^e = 0$. For integral-error feedback, equation 41 is used instead of equation 40, and the unforced error dynamics are described by

$$\begin{aligned}&\dot{\omega}^e + c_1\omega^e + 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e \\ &+2c_i(T^{eT} + \varepsilon^e\varepsilon^{eT}/\eta^e)\int \varepsilon^e dt = 0\end{aligned} \tag{44}$$

away from the singularity at $\eta^e = 0$. Although equations 43 and 44 appear nonlinear, away from the singularity at $\eta^e = 0$ they are equivalent to equations 40 and 41, respectively, which are linear in ε^e.

Control Law

To realize arbitrary unforced error dynamics, equations 21, 33, and 34, along with equations 35 and 36 are used to solve for the required angular acceleration. The resulting closed-loop angular acceleration command is

$$\begin{aligned}\dot{\omega}^* = &R^e\dot{\omega}^c + \omega \times \omega^e + 2(T^{eT} + \varepsilon^e\varepsilon^{eT}/\eta^e)\ddot{\varepsilon}^e \\ &+(\omega^{eT}\omega^e/4)\varepsilon^e/\eta^e\end{aligned} \tag{45}$$

where $\ddot{\varepsilon}^e = f(\varepsilon^e, \dot{\varepsilon}^e, t)$ represents the desired error dynamics, with $\dot{\varepsilon}^e = T^e\omega^e/2$. This is singular at $\eta^e = 0$ or $\phi^e = \pi$. By substituting this into equation 27, the attitude control law is determined to be

$$M^* - \dot{h}^* = J\dot{\omega}^* + \omega \times (J\omega + h) \tag{46}$$

Because the control law specifies only the combined term $M^* - \dot{h}^*$, the individual contributions of each term must be specified according to engineering considerations outside the scope of this paper.

To realize the linear, time-invariant, unforced error dynamics described by equation 40, equation 36 is substituted into equation 43 and the required angular acceleration is solved for. The resulting closed-loop angular acceleration command is then

$$\dot{\omega}^* = R^e\dot{\omega}^c + \omega \times \omega^e - c_1\omega^e - 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e \quad (47)$$

A schematic diagram of the resulting control law, along with the rotational dynamics, is shown in Figure 2. Notice that the sign ambiguity of the Euler parameters is taken care of automatically by this control laws because ε^e/η^e is the same even if β^e is negated. The variable ε/η is equivalent to the Gibbs vector, another minimal form of attitude parameterization that has an obvious singularity at $\eta = 0$. Equation 44 can be used instead of equation 43 for integral-error feedback. The closed-loop angular acceleration command is then

$$\dot{\omega}^* = R^e\dot{\omega}^c + \omega \times \omega^e - c_1\omega^e - 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e$$
$$-2c_i(T^{eT} + \varepsilon^e\varepsilon^{eT}/\eta^e)\int \varepsilon^e dt \quad (48)$$

The control law has been expressed in terms of the error Euler parameters, which are not directly measurable but can be determined from any attitude measurement.

The form of the control law makes the damping and the feedback loop bandwidth easy to set by pole placement. The coefficients from equations 40 and 41 appear directly in equations 47 and 48, respectively. Such simplicity is notably absent in some other methods. In optimal control theory, for example, the basic design parameters (weighting matrices) must be processed by a complicated algorithm to determine the feedback gains (yet this is hardly the most serious objection to most applications of optimal control).

For stability, the poles must of course be in the left half of the s plane. For the tightest possible tracking, the poles should be placed as far into the left half of the s plane as possible without risking instability due to discretization, delays, or unmodeled moment-generation dynamics. Of course, if the tightest possible tracking is not required, it may be wise to place the poles nearer to the origin in order to slow the response and thereby attenuate actuator stress, energy consumption, or excitation of structural vibration modes, for example.

Because the control law is intended for tracking of dynamic commands, it accepts feedforward commands. The feedforward commands improve transient performance substantially by forcing the controller to respond instantly to the commands rather than merely letting it react to the errors. For regulation to a static attitude command, however, or for raw attitude commands that consist of a discontinuous series of rest states, the feedforward commands can simply be deactivated by setting $\omega^c \equiv \dot{\omega}^c \equiv 0$. This will be referred to as feedback-only mode.

Although the control law realizes linear unforced error dynamics, the linearity is not global. The topology of the attitude state space is such that global linearity is mathematically impossible to realize. Thus the control law becomes singular when $\eta^e = \cos(\phi^e/2) = 0$ or $\phi^e = \pi$ rad about any eigenaxis. If the commands are physically achievable without saturating the actuators, this is of no practical concern because the attitude error angle should never approach π rad. The inevitable singularity is essentially as far out of the way as possible.

In the case of initial attitude acquisition from a random state, or in case the commands are so untrackable that the error angle approaches π rad, the simple heuristic tactic of restricting the absolute value of η^e in the control law to some minimim value $\eta^e_{\text{min}} \approx 0.1$ prevents division by zero while the vehicle recovers. This means that for error angles greater than $2\cos^{-1}\eta^e_{\text{min}}$ the unforced error dynamics will be nonlinear. Stability in this nonlinear region has not been mathematically analyzed, but simulation results have yet to show any sign of instability even though many degenerate cases have been tested.

Comparison with Other Control Laws

The Dwyer attitude control law [4, 5] can be expressed in terms of our notation by replacing equation 47 with $\dot{\omega}^* = 2T^{-1}[\ddot{\varepsilon}^c - a_1(\dot{\varepsilon} - \dot{\varepsilon}^c) - a_0(\varepsilon - \varepsilon^c) + (\omega^T\omega/4)\varepsilon]$. This realizes the linear error dynamics described by equation 40, but with the attitude error inappropriately defined in terms of vector algebra as $\varepsilon^e = \varepsilon - \varepsilon^c$, which has no geometric meaning. Also, this control law is singular at any *attitude* (not attitude *error*!) corresponding to an angle of π rad about any eigenaxis, because then the matrix T is singular. This is essentially unacceptable in practice.

The Meyer attitude control law [1, 2] can be expressed in terms of our notation by replacing equation 47 with $\dot{\omega}^* = R^e\dot{\omega}^c + \omega \times \omega^e - c_1\omega^e - c_0\gamma^e$. It has been determined in this study that this control law produces nonlinear unforced error dynamics described by $\ddot{\varepsilon}^e + c_1\dot{\varepsilon}^e + (c_0(\eta^e)^2 - \omega^{eT}\omega^e/4)\varepsilon^e = 0$. For the sake of comparison, suppose that the Wie attitude control law [7] was extended to track dynamic commands. It could then be expressed in terms of our notation by replacing equation 47 with $\dot{\omega}^* = R^e\dot{\omega}^c + \omega \times \omega^e - c_1\omega^e - 2c_0\varepsilon^e$. It has been determined in this study that this control law would then produce nonlinear unforced error dynamics described by $\ddot{\varepsilon}^e + c_1\dot{\varepsilon}^e + (c_0\eta^e - \omega^{eT}\omega^e/4)\varepsilon^e = 0$. In each case the unforced error dynamics reduce to the linear form of equation 40 for small errors, because then $\eta^e \approx 1$ and $\omega^e \approx 0$. The control law proposed here, however, realizes exact linearity even if the errors become large, as long as they do not approach π rad.

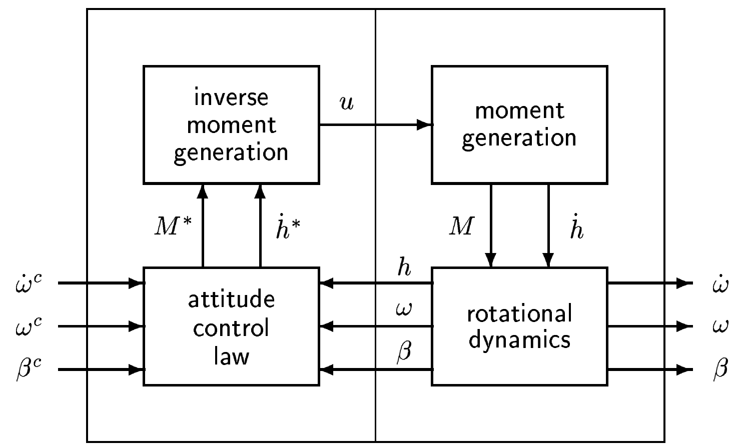
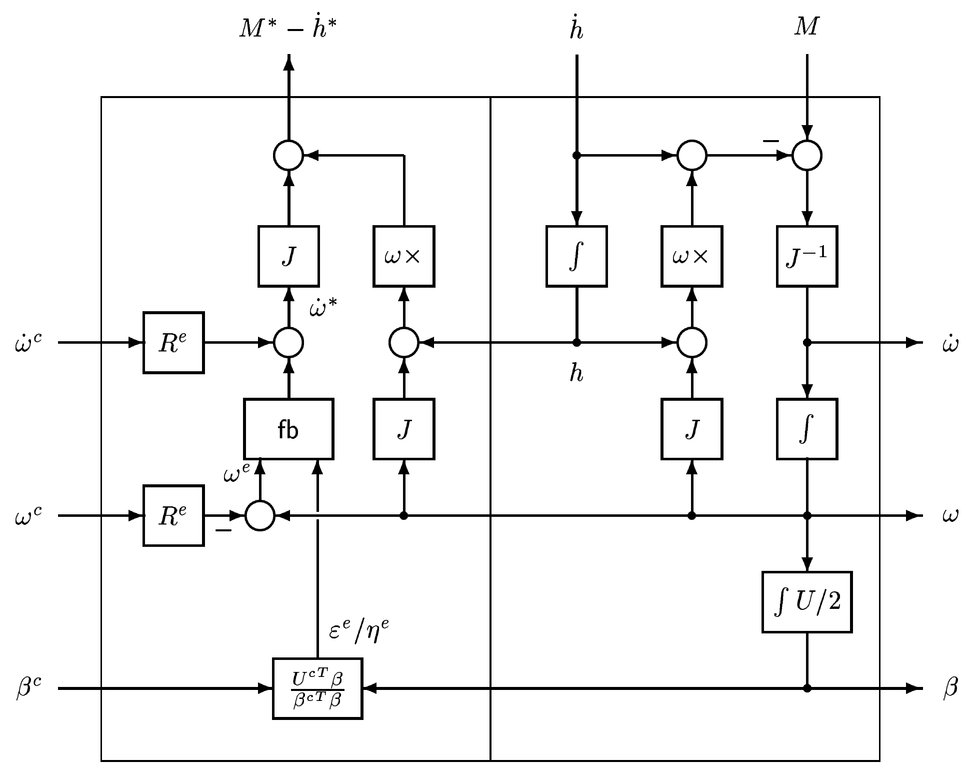


Figure 1: Simplified attitude controller (left half) and vehicle (right half)



$$\mathsf{fb} \equiv \omega \times \omega^e - c_1\omega^e - 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e$$

Figure 2: Attitude control law (left half) and vehicle rotational dynamics (right half)

Moment Generation

The moment generation problem is the problem of determining the actual controls necessary to realize the closed-loop moment command. (For convenience, the word *moment* will often be used to refer to the combined moment and momentum transfer.) In general, the moment generation process is dynamic, involving lags and delays. If the moment generation dynamics are much faster than the unforced error dynamics, however, they can be approximated as static. This is equivalent to a time-scale separation or a singular perturbation approximation.

The poles of the error dynamics can always be placed as close as necessary to the origin of the s plane to make the time-scale separation reasonable. Then the moment generation statics can be canceled, to the extent that that they are known, by a static inverse-moment-generation map. If the poles have to be placed too close to the origin, control accuracy may be inadequate, of course. Then a dynamic inverse-moment-generation model may be feasible, but that may require additional measurements of actuator states, and it is outside the scope of this paper.

The existence of a unique solution to the moment generation problem is not necessary. If multiple solutions exist, then one must be selected based on engineering considerations. If no solution exists, then the commands are physically unachievable and the best available nonsolution must be selected until the vehicle recovers from actuator saturation. If the best available nonsolution is unacceptable, then the problem is outside the scope of control theory. Whereas the attitude control law is common to any rigid vehicle, the moment generation problem is specific to each application and is therefore outside the scope of this paper.

In principle, all knowledge about external moments can be used directly. For aircraft, this includes all theoretical, computational, and experimental knowledge of aerodynamics. It also includes knowledge about the gravity-gradient torque acting on an orbiting spacecraft, for example, and the tether torque acting on a tethered spacecraft. (The tether force could be measured, if necessary, by a three-axis load cell at the attachment point.)

Forced Error Dynamics

The closed-loop moment commands cannot be realized exactly in practice because: 1) the moment generation problem cannot be solved exactly due to modeling errors; 2) unpredictable disturbances will impinge on the vehicle; and 3) the actuators will saturate (in position, rate, etc.) if the commands are not physically acheivable.

Let $\Delta M - \Delta \dot{h}$ is the moment generation error, where $\Delta M = M - M^*$ and $\Delta \dot{h} = \dot{h} - \dot{h}^*$. For the unforced error dynamics described by equations 40 and 43, the corresponding forced error dynamics are described by

$$\dot{\omega}^e + c_1\omega^e + 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e = J^{-1}(\Delta M - \Delta \dot{h}) \quad (49)$$

away from the singularity at $\eta^e = 0$ or, equivalently,

$$\ddot{\varepsilon}^e + c_1\dot{\varepsilon}^e + c_0\varepsilon^e = T^e J^{-1}(\Delta M - \Delta \dot{h}) \quad (50)$$

Although the unforced error dynamics are linear, the forced error dynamics are nonlinear because the factor T^e on the right side of equation 50 is a function of ε^e, and $\Delta M - \Delta \dot{h}$ could also be a function of ε^e and its derivatives. As for any truly nonlinear control problem, linear forced error dynamics are mathematically impossible to realize.

The left side of equation 50 is linear and, because the singular values of T^e are 1, 1, and $|\eta^e|$, where $0 \leq |\eta^e| \leq 1$, the right side of equation 50 is bounded in magnitude by the right side of equation 49. Therefore, the attitude error can be forced into any bounded region of the origin, and trapped in that region, by restricting the moment generation error, and the initial attitude and angular rate errors, to some appropriate nonzero bounds. Those bounds can be found by Lyupanov stability analysis, for example, but such analysis is specific to each application.

This is the best general stability result that could possibly apply to the forced error dynamics for *any* attitude control law. Bounded-input/bounded-output stability (defined such that any bounded input causes a bounded output) doesn't apply because the attitude error is mathematically bounded by definition according to $\| \varepsilon^e \| \leq 1$ and $\phi^e \leq \pi$ rad. Alternatively, if an attitude error angle of π rad (or any other attitude error) is arbitrarily defined to be unbounded, then bounded-input/bounded-output stability is impossible to achieve, because then some bounded error forcing function can always be found that will drive the attitude error unbounded.

For zero attitude error, $T^e = I$, and for small errors, $T^e T^e \approx R^{eT}$ according to equation 23. For small errors, T^e is therefore approximately an orthogonal rotation matrix corresponding to half of the negative of the error angle about the error eigenaxis. For small errors, therefore, premultiplication of $J^{-1}(\Delta M - \Delta \dot{h})$ by T^e corresponds physically to a rotation about the error eigenaxis by half of the rotation angle from the *actual* body frame to the *commanded* body frame.

Simulation

Methods

A controller based on the proposed control law was tested by numerical simulation. A simulation frame rate of 1 kHz was used with trapezoidal integration to approximate the continuous rotational dynamics, and a controller sampling rate of 100 Hz was used. Four-byte arithmetic was used throughout. The inertia tensor used for the simulation results to be presented was $J = \text{diag}\ (2, 2, 3)\cdot 1000$ kg-m^2. The two poles of the error dynamics were each placed at $s = -2.0$ rad/sec.

Moment generation dynamics were neglected in the controller design, but were modeled in the simulation as a first-order lag. The following error sources were simulated: moment disturbances, static and dynamic moment generation errors, inertial measurement errors, command-transducer noise, discretization errors, and quantization errors. The intention was not to accurately model a specific vehicle, or to accurately model error sources, but simply to test the robustness of the new approach to attitude control. Momentum transfer was not used, nor was integral-error feedback.

Static moment generation errors were modeled by simply adding a random error to the moment commands to obtain the applied moment. The random error was modeled as a combination of an arbitrary scale factor, bias, and random noise. The moment generation errors have the same effect as disturbance moments. Also, the scale-factor error mimics roughly the effect of an incorrectly known inertia tensor. Actuator saturation limits were modeled as hard limits on the applied moment in each axis. Dynamic moment generation errors were modeled as lags and delays. The lags were modeled as first-order lags on the applied moments in each axis, and were discretized at the basic simulation frame rate by pole-zero mapping. The final source of error was measurement or estimation error in the inertial measurement unit, modeled as a bias and random noise in each axis of the angular rate and the attitude.

A simple feedforward command generator was designed to accept raw attitude commands in terms of Euler angles (which were used only for ease of visualization). Because noise and quantization error in the raw commands are of particular concern for the case of real-time command generation, random white noise was added to the raw attitude commands to simulate a noisy analog command transducer. The raw commands are then quantized at a specified number of bits to simulate the operation of an analog-to-digital converter. The feedforward command generator prefiltered the raw commands in a second-order linear low-pass filter that was designed by pole-zero mapping to have two fast poles at $s = -50$ rad/sec. The prefilter attenuates the transducer noise and assures a twice-differentiable attitude command signal while introducing only a slight lag.

Table 1: Simulation Error Parameters (each axis)

parameter	value
moment saturation limits	± 2 kN-m
moment lag time constant	100 msec
moment time delay	100 msec
command noise	0.2 deg
command quantization	12 bits
command quantization inc	0.09 deg
moment bias error	100 N-m
moment scale factor error	0.05
moment noise error	50 N-m
attitude estimation noise	0.002 rad
angular-rate meas noise	0.002 rad/sec
attitude estimation bias	0.005 rad
angular-rate meas bias	0.005 rad/sec

Results

Figures 3 and 4 are plots of simulation results for initial attitude acquisition for a tumbling spacecraft. The initial attitude error is 180 deg and the initial angular rate error is 10 rad/sec, which are about as severe as can be expected, both about the first principle axis. This case reduces essentially to a one-dimensional problem (for simplicity of plotting), but for cases (not shown) with multi-axis initial errors the response was fundamentally similar. The error models were inactive, but cases were run both with and without moment saturation limits. The moment saturation limits were 2 kN-m, which corresponds to angular acceleration saturation limits of 1 rad/sec^2. Figure 3 shows the time responses. With saturation inactive, the reference attitude was acquired very gracefully in about 5 sec. For the more realistic case with moment saturation limits, the vehicle spun around for about 5 revolutions before coming to rest, but nevertheless came smoothly to rest in about 17 sec. Figure 4 shows the corresponding phase-plane diagram. The phase-plane curve starts at the point (1,0) and moves clockwise. (The first Euler parameter rate is initially zero, even though the angular rate is high, because the initial attitude is π rad about the first axis.) Without saturation, the phase-plane curve heads quickly to the origin. With saturation, it goes around in an oval of decreasing size in the rate dimension until finally the momentum is low enough that the reference attitude is acquired. This illustrates that the control law works exceptionally well for initial attitude acquisition. Thus, a separate strategy is not required for this case as it is in virtually all other approaches.

Figures 5 and 6 show simulation results for rest-to-rest commands in roll angle, with all error models and moment saturation limits inactive. Figure 5 shows the time response for a 135-deg step command in roll angle. Feedback-only mode is used because it is appropriate for step commands. The response is as good as can be expected. Although the response is linear in the Euler parameters, it is nonlinear in roll angle. Figure 6 shows the response for a rest-to-rest roll command of 135 deg in the form of a fifth-order spline function, which has a bounded second derivative. The response tracks the command so closely that it is indistinguishable from it on the plot, as expected. The feedback-only response, on the other hand, lags the command by almost a full second. This illustrates that the feedforward commands improve transient performance substantially by forcing the controller to respond instantly to the commands rather than merely letting it react to the error.

Figures 7 and 8 show the simulation results for rest-to-rest multiaxis commands in terms of Euler angles, with the error models and saturation limits described in the previous subsection now active. The parameters of the error models and saturation limits are summarized in Table 1. In each case, simultaneous rest-to-rest commands of 30 deg over 4 sec in each of the Euler angles are given. In Figure 7 the commands are in the form of linear ramp functions, which do not have bounded second derivatives; in Figure 8 they are in the form of fifth-order spline functions, which have bounded second derivatives. These raw commands were fed through the second-order linear prefilter described in the previous subsection. The prefilter attenuates the command transducer noise and assures bounded second derivatives for the feedforward command generator. In each run the vehicle follows the commands very closely, with a slight lag due to the command prefiltering and the moment generation lag and delay. The response is particularly impressive in Figure 7 because the control law saturated the actuators by factors of 2.0 and 2.5 in the pitch and yaw axes, respectively. The slight overshoots due to saturation at the corners of the ramp functions are apparent. The slight offset at steady state is due to the simulated attitude measurement (or estimation) bias. These plots demonstrate the robustness of the proposed control method.

Conclusion

An attitude control law has been derived for the first time to realize linear unforced error dynamics with the attitude error properly defined in terms of rotation group algebra (rather than vector algebra). Possible future research topics to extend this new approach include derivation of a dual estimation algorithm, theoretical analysis of robustness to measurement error, adaptation to vehicles with on/off actuators and actuators with slow dynamics, and adaptation to flexible structures.

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Appendix: Derivation Details

The following derivation details have made use of the rotational kinematic relations expressed in equation 14, the eigenvalue property of T expressed in equation 22, the relationship between T and R expressed in equation 23, and the unit-norm constraint of the Euler parameters.

Expressions for the time derivatives of the minimal error Euler parameters were derived as follows:

$$\begin{aligned}
\varepsilon^e &= U^{cT}\beta \\
\dot{\varepsilon}^e &= U^{cT}\dot{\beta} + \dot{U}^{cT}\beta = U^{cT}\dot{\beta} - U^T\dot{\beta}^c \\
&= U^{cT}U\omega/2 - U^T U^c\omega^c/2 \\
&= T^e\omega/2 - T^{eT}\omega^c/2 \\
&= T^e(\omega - T^{e-1}T^{eT}\omega^c)/2 = T^e(\omega - R^e\omega^c)/2 \\
&= T^e\omega^e/2 \\
\ddot{\varepsilon}^e &= T^e\dot{\omega}^e/2 + \dot{T}^e\omega^e/2 \\
&= T^e\dot{\omega}^e/2 + \dot{\varepsilon}^e \times \omega^e/2 + \dot{\eta}^e\omega^e/2 \\
&= T^e\dot{\omega}^e/2 + (T^e\omega^e) \times \omega^e/4 - \omega^e\varepsilon^{eT}\dot{\varepsilon}^e/(2\eta^e) \\
&= T^e\dot{\omega}^e/2 + (\varepsilon^e \times \omega^e + \eta^e\omega^e) \times \omega^e/4 - \\
&\quad \omega^e\varepsilon^{eT}T^e\omega^e/(4\eta^e) \\
&= T^e\dot{\omega}^e/2 + (\varepsilon^e \times \omega^e) \times \omega^e/4 - \omega^e\varepsilon^{eT}\omega^e/4 \\
&= T^e\dot{\omega}^e/2 + \omega^e\omega^{eT}\varepsilon^e/4 - \varepsilon^e\omega^{eT}\omega^e/4 - \\
&\quad \omega^e\varepsilon^{eT}\omega^e/4 \\
&= T^e\dot{\omega}^e/2 - \varepsilon^e\omega^{eT}\omega^e/4
\end{aligned}$$

It is interesting to note that the error kinematic relation $\dot{\varepsilon}^e = T^e\omega^e/2$ could have been anticipated because it is of the same form as the actual kinematic relation $\dot{\varepsilon} = T\omega/2$. Note, however, that a definition of the angular rate error, $\omega^e = \omega - R^e\omega^c$, is obtained in the derivation. Whereas ω^c represents the angular rate command resolved into commanded body coordinates, $R^e\omega^c$ represents the angular rate command resolved into actual body coordinates.

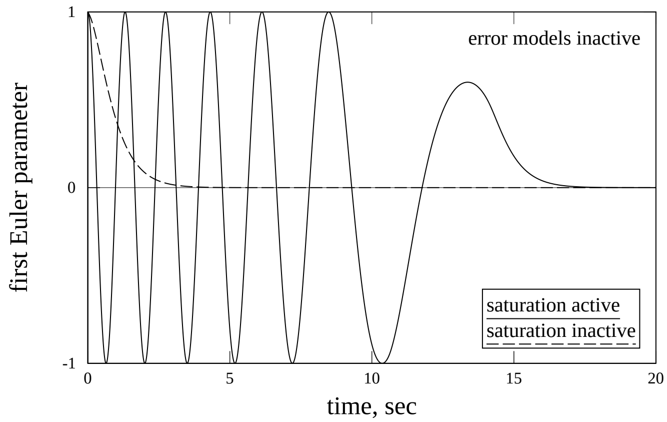

Figure 3: Initial Attitude Acquisition

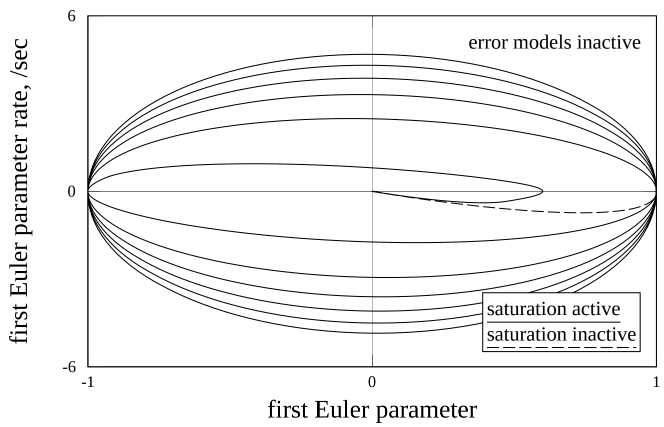

Figure 4: Phase-Plane Diagram For Attitude Acquisition

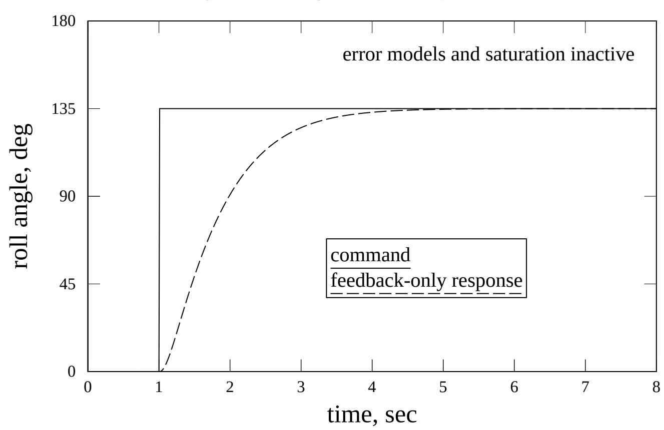

Figure 5: Single-Axis Step Sommand

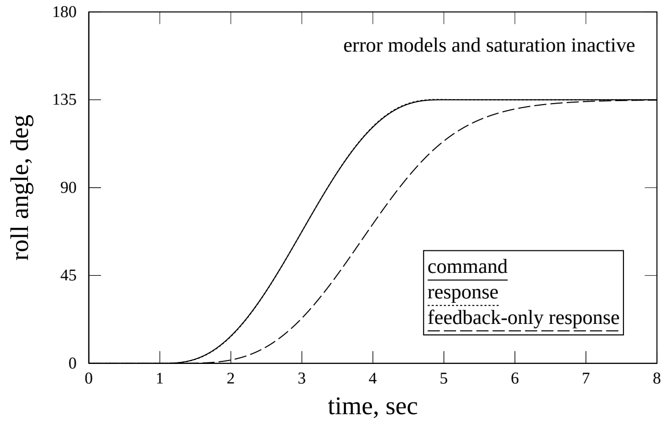

Figure 6: Continuous Single-Axis Command

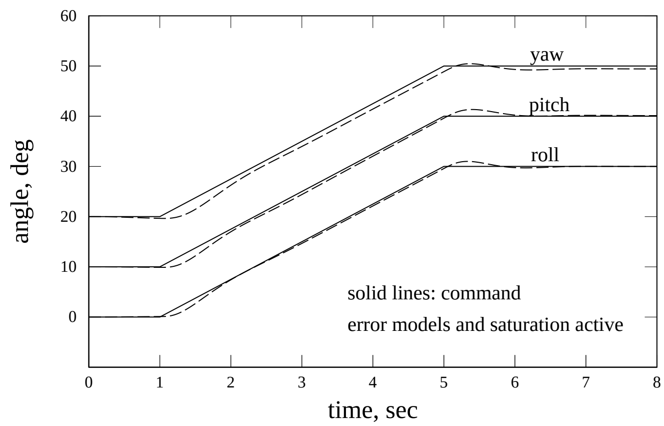

Figure 7: Continuous Multiaxis Command

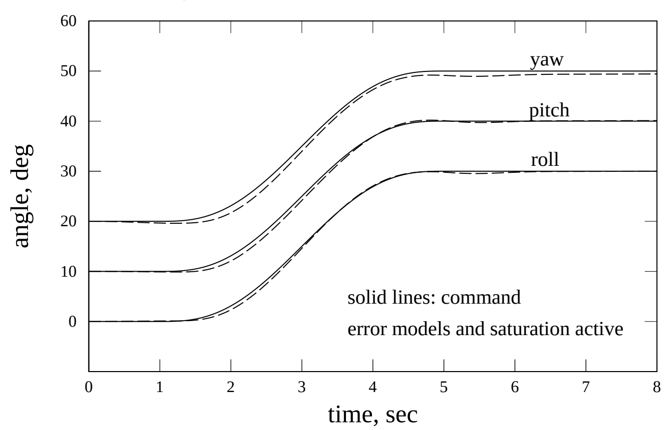

Figure 8: Continuous Multiaxis Command

The following derivation is also needed.

$$
\begin{aligned}
2(T^e)^{-1}(\ddot{\varepsilon}^{\,e} + c_1\dot{\varepsilon}^e + c_0\varepsilon^e) \\
&= (T^{eT} + \varepsilon^e\varepsilon^{eT}/\eta^e)(T^e\dot{\omega}^e - \varepsilon^e\omega^{eT}\omega^e/2 + \\
&\quad c_1T^e\omega^e + 2c_0\varepsilon^e) \\
&= \dot{\omega}^e - T^{eT}\varepsilon^e\omega^{eT}\omega^e - \varepsilon^e\varepsilon^{eT}\varepsilon^e\omega^{eT}\omega^e/\eta^e + \\
&\quad c_1\omega^e + 2c_0(T^{eT}\varepsilon^e + \varepsilon^e\varepsilon^{eT}\varepsilon^e/\eta^e) \\
&= \dot{\omega}^e - \eta^e\varepsilon^e\omega^{eT}\omega^e/2 - \varepsilon^e\varepsilon^{eT}\varepsilon^e\omega^{eT}\omega^e/(2\eta^e) + \\
&\quad c_1\omega^e + 2c_0(\eta^e\varepsilon^e + \varepsilon^e\varepsilon^{eT}\varepsilon^e/\eta^e) \\
&= \dot{\omega}^e - (\eta^{e\,2} + \varepsilon^{eT}\varepsilon^e)\varepsilon^e\omega^{eT}\omega^e/(2\eta^e) + c_1\omega^e + 2c_0 \\
&\quad (\eta^{e\,2} + \varepsilon^{eT}\varepsilon^e)\varepsilon^e/\eta^e \\
&= \dot{\omega}^e - \varepsilon^e\omega^{eT}\omega^e/(2\eta^e) + c_1\omega^e + 2c_0\varepsilon^e/\eta^e \\
&= \dot{\omega}^e + c_1\omega^e + 2(c_0 - \omega^{eT}\omega^e/4)\varepsilon^e/\eta^e
\end{aligned}
$$